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Symmetry groups of Rosenbloom–Tsfasman spaces

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Abstract

Let $\mathbb{F}_q^{m \cdot n}$ be the vector space of $m \cdot n$-tuples over a finite field $\mathbb{F}_q$ and $P = \{1, 2, \ldots, m \cdot n\}$ a poset that is the finite union of m disjoint chains of length n. We consider on $\mathbb{F}_q^{m \cdot n}$ the poset-metric d_P introduced by Rosenbloom and Tsfasman. In this paper, we give a complete description of the symmetry group of the metric space (V, d_P).

Keywords: Poset codes; Poset metrics; Rosenbloom–Tsfasman metric; Symmetries

1. Introduction

One of the main classical problems of the coding theory is to find sets with q^k elements in $\mathbb{F}_q^n$, the space of n-tuples over the finite field $\mathbb{F}_q$, with the largest minimum distance possible. There are many possible metrics that can be defined in $\mathbb{F}_q^n$, the most common ones are the Hamming and Lee metrics.

In 1987 Harald Niederreiter generalized the classical problem of coding theory (see [7]). Brualdi, Graves and Lawrence (see [1]) also provided in 1995 a wider situation for the above problem: using partially ordered sets (posets) and defining the concept of poset-codes, they started to study codes with a poset-metric. This has been a fruitful approach, since many new perfect codes have been found with such poset metrics [3,5].

A particular instance of poset-codes and spaces are the spaces introduced by Rosenbloom and Tsfasman in 1997 [9], where the posets taken into consideration have a finite number of disjoint chains of equal size, that is, are isomorphic to a order $P = P_1 \cup P_2 \cup \cdots \cup P_m$ such that

$$P_{i+1} = \{in + 1, in + 2, \ldots, (i + 1)\, n\}$$

and

$$in + 1 < in + 2 < \cdots < (i + 1)\, n$$

are the only strict comparabilities for each $i \in \{0, 1, \ldots, m - 1\}$.

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The description of linear symmetries of a poset-space started with the study of particular poset spaces (as Lee's work on Rosenbloom–Tsfasman spaces [6]; Cho and Kim's work on crown spaces [2]; Kim's work on weak spaces [4]), until Panek, Firer, Kim and Hyun [8] gave a full description of the groups of linear symmetries of a poset space. In this work, we describe the symmetry group (not necessarily linear ones) of a poset-space that is a product of Rosenbloom–Tsfasman spaces. We hope that, as in the case of linear symmetries, the particular cases can enlighten the general one.

In the Section 2, we introduce briefly the main concepts and definitions used in this work. In the Section 3, we study the simple, but inspiring, case of posets determining a single chain (Theorem 3.1) and finally, in the last two sections, we describe the symmetry groups of Rosenbloom–Tsfasman spaces and product of such spaces (Theorems 4.1 and 5.1).

2. Poset metric spaces

Let $[n] := \{1, 2, \ldots, n\}$ be a finite set with n elements and let $\leq$ be a partial order on $[n]$. We call the pair $P := ([n], \leq)$ a *poset* and say that k is *smaller than* j if $k \leq j$ and $k \neq j$. An *ideal* in $([n], \leq)$ is a subset $I \subseteq [n]$ that contains every element that is smaller than some of its elements, i.e., if $j \in I$ and $k \leq j$ then $k \in I$. Given a subset $X \subseteq [n]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the *ideal generated by* X. An order on the finite set $[n]$ is called a *linear order* or a *chain* if every two elements are comparable, that is, given $i, j \in [n]$ we have that either $i \leq j$ or $j \leq i$. In this case, n is said to be *the length* of the chain and the set can be labeled in such a way that $i_1 < i_2 < \cdots < i_n$. For the simplicity of the notation, in this situation we will always assume that the order P is defined as $1 < 2 < \cdots < n$.

Given an order $P = ([n], \leq)$ and $i, j \in [n]$, we say that $i_0, i_1, \ldots, i_k$ is a *gallery (path) connecting* i and j if $i = i_0$, $j = i_k$ and for every $1 \leq l \leq k$, either $i_{l-1} \leq i_l$ or $i_l \leq i_{l-1}$. We say that i and j *can be connected* if there is a gallery connecting them. Since the concatenation of galleries is still a gallery, the above property defines an equivalence relation on $[n]$ and each equivalence class is called a *connected component* of P.

We note that every connected component of P (as every subset of $[n]$) is by itself a poset (with the induced order).

Let q be a power of a prime, $\mathbb{F}_q$ be the finite field of q elements and $V := \mathbb{F}_q^n$ the n-dimensional vector space of n-tuples over $\mathbb{F}_q$. Given $v \in V$ we denote by $v = (v_1, v_2, \ldots, v_n)$ its coordinates. The poset $([n], \leq)$ induces a metric $d_P(\cdot, \cdot)$ on V, called *a poset metric* [1], defined as

$$d_P(u, v) := |\langle \operatorname{supp}(u - v) \rangle|$$

where $\operatorname{supp}(w) := \{i \in [n] : w_i \neq 0\}$ is the *support of the vector* w, and $|X|$ is the cardinality of the set X. The pair (V, d_P) is a metric space and where no ambiguity may rise, we say it is a *poset space*. The distance $\omega_P(v) := d_P(0, v)$ is called the P-*weight* of v.

A *symmetry of* (V, d_P) is a bijection $T : V \longrightarrow V$ that preserves distance:

$$d_P(u, v) = d_P(T(u), T(v))$$

for all $u, v \in V$.

The set G of symmetries of (V, d_P) is a group with the natural operation of composition of functions, and we call it the (full) *symmetry group* of (V, d_P).

The description of the full symmetry group may be of help in the study of non-linear codes. Besides other applications, linear symmetries are used to divide linear codes in equivalence classes, since they take subspace into subspace and preserve dimension and minimum distance. Symmetries, in general, may take linear codes onto non-linear ones, but preserve all metric data, such as minimal distance and weight of a code and also the generalized Wei weights, capacity of error correction and number of elements. So it is just natural to call two non-linear codes *equivalent* if one is the image of the other under a symmetry.

In [8] the group of linear symmetries of a poset space is characterized, for any given poset. In this work we will describe the full symmetry group of an important class of poset spaces, namely, those induced by posets such that every connected component is a chain. This class includes spaces over chains (linear orders) and the Rosenbloom–Tsfasman spaces, where the associated poset consists of a finite disjoint union of chains of equal length.

3. Symmetries of chain spaces

Let $P = ([n], \leq)$ be a chain of length n, $\mathbb{F}_q$ a finite field and $V := \mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$ with the poset metric d_P. In this section we will describe the full symmetry group of the poset space V. This description will be used in the next section to describe the symmetry group of a Rosenbloom–Tsfasman space. All posets in this section will be linear posets.

We note that, given $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in the poset space V,

$$d_P(u, v) = \max\{i : u_i \neq v_i\}.$$

For each $i \in \{1, 2, \ldots, n\}$, let $F_i : \mathbb{F}_q^{n-i+1} \to \mathbb{F}_q$ be a map that is a bijection with respect to the first coordinate, that is, given $v_{i+1}, \ldots, v_n \in \mathbb{F}_q$, the map $\widetilde{F}_{v_{i+1},\ldots,v_n} : \mathbb{F}_q \to \mathbb{F}_q$ defined by

$$\widetilde{F}_{v_{i+1},\ldots,v_n}(v_i) = F_i(v_i, v_{i+1}, \ldots, v_n)$$

is a bijection. Given such a family, we define a map $T_{(F_1,F_2,\ldots,F_n)} : V \to V$ by

$$T_{(F_1,F_2,\ldots,F_n)}(v_1, \ldots, v_n) := (F_1(v_1, \ldots, v_n), F_2(v_2, \ldots, v_n), \ldots, F_n(v_n)).$$

Lemma 3.1. *Let $P = ([n], \leq)$ be a linear poset, with $1 < 2 < \cdots < n$. Let V be the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ endowed with the poset metric induced by the poset P. The map $T_{(F_1,F_2,\ldots,F_n)} : V \longrightarrow V$ is a symmetry of V.*

Proof. Given $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in V$, let $l = d_P(u, v) = \max\{i : u_i \neq v_i\}$. Since each $F_i : \mathbb{F}_q^{n-i+1} \to \mathbb{F}_q$ is a bijection in relation to the first coordinate, we have that

$$F_l(u_l, u_{l+1}, \ldots, u_n) \neq F_l(v_l, v_{l+1}, \ldots, v_n)$$

and

$$F_t(u_t, u_{t+1}, \ldots, u_n) = F_t(v_t, v_{t+1}, \ldots, v_n)$$

for any $t > l$.

It follows that

$$\begin{aligned} d_P\left(T_{(F_1,\ldots,F_n)}(u), T_{(F_1,\ldots,F_n)}(v)\right) &= \max\{i : F_i(u_i, \ldots, u_n) \neq F_i(v_i, \ldots, v_n)\} \\ &= l \end{aligned}$$

and hence $T_{(F_1,F_2,\ldots,F_n)}$ is distance preserving. Since V is a finite metric space, it follows that $T_{(F_1,F_2,\ldots,F_n)}$ is also a bijection. □

In the previous lemma we attained a large set of symmetries of V. The following lemma shows every symmetry may be expressed in this form.

Lemma 3.2. *Let $P = ([n], \leq)$ be a linear poset, with $1 < 2 < \cdots < n$. Let V be the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ endowed with the poset metric induced by the poset P. Let $T : V \to V$ be a symmetry of V, with a linear poset metric. Then, there are functions $F_i : \mathbb{F}_q^{n-i+1} \to \mathbb{F}_q$ such that*

(i) $T(v_1, v_2, \ldots, v_n) = (F_1(v_1, v_2, \ldots, v_n), F_2(v_2, \ldots, v_n), \ldots, F_n(v_n))$.
(ii) *For every $i \in \{1, \ldots, n\}$ and each $(v_{i+1}, \ldots, v_n) \in \mathbb{F}_q^{n-i+2}$ the function $\widetilde{F}_{v_{i+1},\ldots,v_n} : \mathbb{F}_q \to \mathbb{F}_q$ defined by $\widetilde{F}_{v_{i+1},\ldots,v_n}(v_i) = F_i(v_i, v_{i+1}, \ldots, v_n)$ is a bijection.*

Proof. Let us write

$$T(v_1, v_2, \ldots, v_n) = (T_1(v_1, v_2, \ldots, v_n), \ldots, T_n(v_1, v_2, \ldots, v_n)).$$

We prove first that $T_j(v_1, v_2, \ldots, v_n) = F_j\left(v_j, v_{j+1}, \ldots, v_n\right)$, that is, T_j does not depends on the first $j - 1$ coordinates. In other words, we want to prove that

$$T_j\left(v_1, \ldots, v_{j-1}, v_j, \ldots, v_n\right) = T_j\left(u_1, \ldots, u_{j-1}, v_j, \ldots, v_n\right)$$

regardless of the values of the first $j-1$ coordinates. But

$$d\left(\left(u_1,\dots,u_{j-1},v_j,\dots,v_n\right),\left(v_1,\dots,v_{j-1},v_j,\dots,v_n\right)\right)=\max_i\{i : v_i\neq u_i\}\leq j-1$$

and since T is a symmetry, we find that

$$\begin{aligned}&d\left(T\left(v_1,\dots,v_{j-1},v_j,\dots,v_n\right),T\left(u_1,\dots,u_{j-1},v_j,\dots,v_n\right)\right)\\&\quad=d\left(\left(u_1,\dots,u_{j-1},v_j,\dots,v_n\right),\left(v_1,\dots,v_{j-1},v_j,\dots,v_n\right)\right)\leq j-1\end{aligned}$$

and so

$$T_j\left(v_1,\dots,v_{j-1},v_j,\dots,v_n\right)=T_j\left(u_1,\dots,u_{j-1},v_j,\dots,v_n\right)$$

for any $v_1,\dots,v_{j-1},v_j,\dots,v_n,u_1,\dots,u_{j-1}\in\mathbb{F}_q$. We find that

$$T\left(v_1,v_2,\dots,v_n\right)=\left(F_1\left(v_1,v_2,\dots,v_n\right),F_2\left(v_2,\dots,v_n\right),\dots,F_n\left(v_n\right)\right)$$

and the first statement is proved.

Now we need to prove that each $\widetilde{F}_{v_{i+1},\dots,v_n}$ is a bijection, what is equivalent to prove those maps are injective. Suppose $\widetilde{F}_{v_{i+1},\dots,v_n}$ is not injective, so there are $v_i\neq u_i$ such that

$$\widetilde{F}_{v_{i+1},\dots,v_n}\left(v_i\right)=\widetilde{F}_{v_{i+1},\dots,v_n}(u_i).$$

Considering i minimal with this property, we would have

$$\begin{aligned}i&=d_P\left(\left(v_1,\dots,v_i,\dots,v_n\right),\left(v_1,\dots,u_i,\dots,v_n\right)\right)\\&=d_P\left(T\left(v_1,\dots,v_i,\dots,v_n\right),T\left(v_1,\dots,u_i,\dots,v_n\right)\right)\\&<i\end{aligned}$$

contradicting the assumption that T is a symmetry of V. $\square$

The next theorem follows straightforward from the previous lemmas.

Theorem 3.1. *Let $P=([n],\leq)$ be a linear poset, with $1<2<\cdots<n$. Let V be the n-dimensional vector space $\mathbb{F}_q^n$ over a finite field $\mathbb{F}_q$ endowed with the poset metric induced by the poset P. Then, the group G_n of symmetries of V is the set of all maps $T_{(F_1,F_2,\dots,F_n)}:V\to V$ where $T_{(F_1,F_2,\dots,F_n)}$ is defined as in Lemma 3.1.*

We remark that $T_{(F_1,F_2,\dots,F_n)}:V\to V$, where $T_{(F_1,F_2,\dots,F_n)}$ is defined as in Lemma 3.1, is linear if and only if each F_i is linear, that is, $F_i\left(x_i,\dots,x_n\right)=a_{ii}x_i+\cdots+a_{in}x_n$. Bijectivity with respect to the first coordinate yields $a_{ii}\neq 0$. Hence, the matrix of $T_{(F_1,F_2,\dots,F_n)}$ with respect to the canonical basis $\{e_1,\dots,e_n\}$ of $\mathbb{F}_q^n$ is an invertible upper triangular matrix (as in [6], Theorem 1):

$$T_{(F_1,F_2,\dots,F_n)}\left(e_j\right)=\sum_{i=1}^{n}F_i\left(e_j\right)e_i=\sum_{i=1}^{j}a_{ij}e_i.$$

We recall that in Lemma 3.2, considering the linear poset $([n+1],\leq)$ we have that $\widetilde{F}_{v_2,\dots,v_{n+1}}\left(v_1\right)=F_1\left(v_1,v_2,\dots,v_{n+1}\right)$ is a bijection, hence a permutation, of $\mathbb{F}_q$, for each $\left(v_2,\dots,v_{n+1}\right)\in\mathbb{F}_q^n$. If S_q denotes the symmetric group of permutations of a set with q elements, since $\mathbb{F}_q^n$ has q^n elements, we can identify the group of functions $F:\mathbb{F}_q^{n+1}\to\mathbb{F}_q$ such that $\widetilde{F}_{v_2,\dots,v_{n+1}}$ is a permutation of $\mathbb{F}_q$ with the power group $\left(S_q\right)^{q^n}$. With these notations we have the following result:

Corollary 3.1. *Let $P=([n],\leq)$ be a linear poset, with $1<2<\cdots<n$. Let $\mathbb{F}_q^n$ be an n-dimensional vector space over a finite field $\mathbb{F}_q$ endowed with the poset metric induced by the poset P. Let G_n be the symmetry group of $\mathbb{F}_q^n$. Then the group G_{n+1}, the symmetry group of $\mathbb{F}_q^{n+1}$ with respect to the poset metric defined by $([n+1],\leq)$, has a semi-direct product structure $G_{n+1}=(S_q)^{q^n}\rtimes G_n$, where q is the cardinality of the field $\mathbb{F}_q$ and S_q denotes the symmetric group of permutations of a set with q elements.*

Proof. In order to simplify notation, we will denote the elements of $(S_q)^{q^n}$ by $(\pi_{(a_1,\ldots,a_n)})$. The group G_n acts on $\mathbb{F}_q^{n+1}$ by

$$T\,(x_1,\ldots,x_{n+1}) = (x_1, T\,(x_2,\ldots,x_{n+1}))$$

and $(S_q)^{q^n}$ acts by

$$\pi_{(a_1,\ldots,a_n)}\,(x_1,\ldots,x_{n+1}) = \left(\pi_{(x_2,\ldots,x_{n+1})}\,(x_1)\,, x_2,\ldots,x_{n+1}\right).$$

Both groups act as groups of symmetries and both act faithfully; therefore these actions establish isomorphisms of these groups with subgroups of G_{n+1}, and we identify $(S_q)^{q^n}$ and G_n with their counterparts $H \cong (S_q)^{q^n}$ and $K \cong G_n$ in G_{n+1}. From the actions defined above, it is easy to see that

$$H = \{T \in G_{n+1};\ T = (F_1, x_2,\ldots,x_{n+1})\}$$

and

$$K = \{T \in G_{n+1};\ T = (x_1, F_2, F_3,\ldots,F_{n+1})\}$$

where each F_i satisfies the conditions of Lemma 3.1. Clearly, $G_{n+1} = HK$, because each symmetry of $\mathbb{F}_q^{n+1}$ is a composition $T_1 \circ T_2$ with $T_1 \in H$ and $T_2 \in K$. We claim that G_{n+1} is a semi-direct product of H by K.

Let $L \in H \cap K$. Since $L \in H$, $L(x_1, x_2,\ldots,x_{n+1}) = (x_1', x_2,\ldots,x_{n+1})$ and, since L is also in K, $x_1' = x_1$. Hence $L = id$ and the groups H and K intersect trivially.

We prove now that H is a normal subgroup of G_{n+1}. In fact, since $G_{n+1} = HK$, it suffices to check that $THT^{-1} \subset H$ for each $T \in K$. Let $(\pi_{(a_1,\ldots,a_n)}) \in H$ and $T \in K$. If $(x_1,\ldots,x_{n+1}) \in \mathbb{F}_q^{n+1}$ then

$$\begin{aligned}
\left(T \circ \pi_{(a_1,\ldots,a_n)} \circ T^{-1}\right)(x_1,\ldots,x_{n+1}) &= \left(T \circ \pi_{(a_1,\ldots,a_n)}\right)\left(x_1, T^{-1}\,(x_2,\ldots,x_{n+1})\right) \\
&= T\left(\pi_{T^{-1}(x_2,\ldots,x_{n+1})}\,(x_1)\,, T^{-1}\,(x_2,\ldots,x_{n+1})\right) \\
&= \left(\pi_{T^{-1}(x_2,\ldots,x_{n+1})}\,(x_1)\,, T \circ T^{-1}\,(x_2,\ldots,x_{n+1})\right) \\
&= \left(\pi_{T^{-1}(x_2,\ldots,x_{n+1})}\,(x_1)\,, x_2,\ldots,x_{n+1}\right) \\
&= \pi_{T^{-1}(x_2,\ldots,x_{n+1})}(x_1, x_2,\ldots,x_{n+1}).
\end{aligned}$$

This shows that H is a normal subgroup of G_{n+1} and that $G_{n+1} \cong H \rtimes K$. Using the aforementioned isomorphisms involving H and K we conclude that $G_{n+1} \cong (S_q)^{q^n} \rtimes G_n$. $\square$

Remark 3.1. It is easy to see that this semi-direct product in the preceding Corollary is not a direct-product. If we consider $T \in S_q$ to be the transposition $(0, 1)$ and define permutations $\pi_0 = (0, 1)$ and $\pi_1 = \mathrm{id}$ we find that

$$\begin{aligned}
\left(\pi_0^{-1} \circ T \circ \pi_0\right)(0, 0) &= \left(\pi_0^{-1} \circ T\right)(\pi_0\,(0)\,, 0) \\
&= \left(\pi_0^{-1} \circ T\right)(1, 0) = \pi_0^{-1}\,(1, T\,(0)) \\
&= \pi_0^{-1}\,(1, 1) = \left(\pi_1^{-1}\,(1)\,, 1\right) = (1, 1)
\end{aligned}$$

and hence $\pi_0^{-1} \circ T \circ \pi_0 \notin G_2$.

4. Symmetries of Rosenbloom–Tsfasman spaces

In this section we consider an order $P = ([m \cdot n], \leq)$ that is the union of m disjoint chains $P_1, P_2,\ldots,P_m$ of order n. We identify the elements of $[m \cdot n]$ with the set of ordered pairs of integers (i, j), with $1 \leq i \leq m$, $1 \leq j \leq n$, where $(i, j) \leq (k, l)$ *iff* $i = k$ and $j \leq_{\mathbb{N}} l$, where $\leq_{\mathbb{N}}$ is just the usual order on $\mathbb{N}$. We denote $P_i = \{(i, j) : 1 \leq j \leq n\}$. Each P_i is a chain and those are the connected components of $([m \cdot n], \leq)$.

Given a finite field $\mathbb{F}_q$ we identify $V := \mathbb{F}_q^{m\cdot n}$ with the set of matrices $\{(v_{ij}) : 1 \le i \le m,\ 1 \le j \le n\}$. The space $V = \mathbb{F}_q^{m\cdot n}$, with the poset metric induced by the order $P = ([m \cdot n], \le)$ is called the *$m \cdot n$-Rosenbloom–Tsfasman space over* $\mathbb{F}_q$. Note that if $n = 1$, then $P = ([m \cdot 1], \le)$ induces just the usual Hamming metric on $\mathbb{F}_q^{m\cdot 1}$. Hence the induced metric from the poset $P = ([m \cdot n], \le)$ can be viewed as a generalization of the Hamming metric. Let $V_i := \{v \in V : \text{supp}(v) \subseteq P_i\}$. It follows that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ and this is called the *canonical decomposition of* V. Given decompositions $u = u_1 + \cdots + u_m$ and $v = v_1 + \cdots + v_m$ with $u_i, v_i \in V_i$, it is well known that

$$d_P(u, v) = \sum_{i=1}^{m} d_P(u_i, v_i) = \sum_{i=1}^{m} d_{P_i}(u_i, v_i)$$

where d_{P_i}, the restriction of d_P to V_i, is a linear poset metric. We note that the restriction of d_P to each V_i turns it into a poset space defined by a linear order, that is, each V_i is isometric to $(\mathbb{F}_q^n, d_{[n]})$ with the metric $d_{[n]}$ determined by the chain $1 < 2 < \cdots < n$. Let G_n be the group of symmetries of $(\mathbb{F}_q^n, d_{[n]})$. The direct product $(G_n)^m$ acts on V in the following manner: given $(T_1, \ldots, T_m) \in (G_n)^m$ and $v \in V$,

$$T(v) = \sum_{i=1}^{m} T_i(v_i)$$

is a symmetry of V.

Lemma 4.1. *Let (V, d_P) be the $m \cdot n$-Rosenbloom–Tsfasman space over $\mathbb{F}_q$ and let G_n be the group of symmetries of $\left(\mathbb{F}_q^n, d_{[n]}\right)$. Given $T_i \in G_n$, with $1 \le i \le m$, the map $T = (T_1, \ldots, T_m)$ defined by*

$$T(v) = \sum_{i=1}^{m} T_i(v_i)$$

is a symmetry of V.

Proof. Given $u, v \in V$, consider the decompositions $u = u_1 + \cdots + u_m$ and $v = v_1 + \cdots + v_m$ with $u_i, v_i \in V_i$. Then,

$$\begin{aligned} d_P\,(T(u), T(v)) &= d_P\left(\sum_{i=1}^{m} T_i(u_i), \sum_{i=1}^{m} T_i(v_i)\right) \\ &= \sum_{i=1}^{m} d_P\,(T_i(u_i), T_i(v_i)) \\ &= \sum_{i=1}^{m} d_P\,(u_i, v_i) \\ &= d_P\,(u, v). \quad \square \end{aligned}$$

Let us consider the canonical decomposition $v = v_1+v_2+\cdots+v_m$ of a vector v in the Rosenbloom–Tsfasman space $V = \mathbb{F}_q^{m\cdot n}$. The symmetric group S_m acts on V permuting its components: given $\sigma \in S_m$ and $v = v_1+v_2+\cdots+v_m \in V$, we define

$$T_\sigma(v) := v_{\sigma(1)} + v_{\sigma(2)} + \cdots + v_{\sigma(m)}$$

Lemma 4.2. *Let (V, d_P) be the $m\cdot n$-Rosenbloom–Tsfasman space over $\mathbb{F}_q$ and let $\sigma \in S_m$, where S_m is the symmetric group. Then, T_σ is a symmetry of (V, d_P).*

Proof. Given $u, v \in V$, we consider their canonical decompositions $u = u_1 + \cdots + u_m$ and $v = v_1 + \cdots + v_m$ with $u_i, v_i \in V_i$. Then,

$$d_P\,(T_\sigma(u), T_\sigma(v)) = d_P\left(\sum_{i=1}^{m} u_{\sigma(i)}, \sum_{i=1}^{m} v_{\sigma(i)}\right)$$

$$= \sum_{i=1}^{m} d_P\left(u_{\sigma(i)}, v_{\sigma(i)}\right)$$
$$= \sum_{i=1}^{m} d_P\left(u_i, v_i\right)$$
$$= d_P(u, v). \quad \square$$

The two previous lemmas assure that the groups $(G_n)^m$ and S_m are both symmetry groups of the Rosenbloom–Tsfasman space $\mathbb{F}_q^{m\cdot n}$, and so is the group $G_{m\cdot n}$ generated by both of them. We identify $(G_n)^m$ and S_m with their images in $G_{m\cdot n}$ and make an abuse of notation, denoting the images in $G_{m\cdot n}$ by the same symbols. With this notation, analogous calculations as those of Corollary 3.1 show that $(G_n)^m \cap S_m = \{id_V\}$ and $\sigma\circ(G_n)^m\circ\sigma^{-1} = (G_n)^m$ for every $\sigma \in S_m$ and we have proved the following:

Proposition 4.1. *Let (V, d_P) be the $m\cdot n$-Rosenbloom–Tsfasman space over $\mathbb{F}_q$. Let $(G_n)^m$ and S_m be the symmetry groups of V as defined in Lemmas 4.1 and 4.2 respectively and let $G_{m\cdot n}$ be the symmetry group of V generated by $(G_n)^m$ and S_m. Then, $G_{m\cdot n}$ has the structure of a semi-direct product $G_{mn} = (G_n)^m \rtimes S_m$.*

We need two more lemmas in order to prove that every symmetry of the $m\cdot n$-Rosenbloom–Tsfasman space V is in $G_{m\cdot n}$, i.e., that $G_{m\cdot n}$ is the group of symmetries of V.

Lemma 4.3. *Let (V, d_P) be the $m\cdot n$-Rosenbloom–Tsfasman space over $\mathbb{F}_q$ and let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ be the canonical decomposition of V. If $T : V \to V$ is a symmetry such that $T(0) = 0$, then, for each index $1 \le i \le m$ there corresponds another index $1 \le j \le m$ such that $T(V_i) = V_j$.*

Proof. In the following we denote the subspace generated by the vectors $v_1, v_2, \ldots, v_s$ by $\mathbb{F}_q v_1 + \mathbb{F}_q v_2 + \cdots + \mathbb{F}_q v_s$. We begin by showing that if u is a vector of P-weight equal to 1, then there is another vector v of equal P-weight such that $T(\mathbb{F}_q u) = \mathbb{F}_q v$. In fact, if $\omega_P(u) = 1$ then $d_P(u, w) = 1$ if and only if $w = \lambda u$ for some $\lambda \in \mathbb{F}_q$.

Since $T(0) = 0$, $T(u)$ is another vector v with $\omega_P(v) = 1$. If $\lambda \in \mathbb{F}_q^* = \mathbb{F}_q - \{0\}$ then $1 = d(u, \lambda u) = d(v, T(\lambda u))$, and $T(\lambda u) = \alpha v$ for some $\alpha \in \mathbb{F}_q^*$. Hence $T(\mathbb{F}_q u) = \mathbb{F}_q v$.

Let $\beta = \{e_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ be the canonical basis of V, where the (i, j)th entry of the matrix $e_{i,j}$ is equal to 1, and all other entries are null. Let

$$V_{s,k} := \mathbb{F}_q e_{s,1} + \mathbb{F}_q e_{s,2} + \cdots + \mathbb{F}_q e_{s,k},$$

with $1 \le s \le m$, $1 \le k \le n$. We note that $V_{s,n} = V_s$. We will prove, by induction on k, that for each s there exists an index l such that

$$T(V_{s,k}) = V_{l,k}$$

for all k, $1 \le k \le m$.

Without loss of generality, let us consider $i = 1$, $P_1 = \{(1, 1), \ldots, (1, n)\}$. Since $\omega_P(e_{1,1}) = 1$, there is a vector $e_{l,1}$ of same weight such that $T(\mathbb{F}_q e_{1,1}) = \mathbb{F}_q e_{l,1}$. Let P_l be the chain that begins at $(l, 1)$, and suppose that $V_{1,k-1}$ is taken by T onto $V_{l,k-1}$.

Let $v = \lambda_1 e_{1,1} + \cdots + \lambda_k e_{1,k}$ and let $T(v) = u_1 + \cdots + u_n$, $u_i \in V_i$; since $T(0) = 0$,

$$\omega_P(v) = \omega_P(T(v)) = \omega_P(u_1) + \cdots + \omega_P(u_n).$$

We will use this to show that $T(v) = u_l$. First suppose that $u_l = 0$. In this case, $\omega_P(v) = \sum_{j\ne l} \omega_P(u_j)$ and therefore

$$k = d_P(e_{1,1}, v) = d_P(T(e_{1,1}), T(v)) = \sum_{j\ne l} \omega_P(u_j) + \omega_P(e_{l,1}) = k + 1,$$

a contradiction. Hence $u_l \ne 0$.

Let $u_l = \alpha_1 e_{l,1} + \cdots \alpha_t e_{l,t}$, and suppose now there is another summand $u_i \ne 0$. Then $k = \sum_j \omega_P(u_j) > \omega_P(u_l)$ and therefore $t < k$. By the induction hypothesis, $T^{-1}(u_l)$ is a vector in $V_{1,k-1}$ with $\omega_P(T^{-1}(u_l)) < k$. Hence

$$k = d_P(T^{-1}(u_l), v) = d_P(u_l, T(v)) = \sum_{j\ne l} \omega_P(u_j) < k,$$

again a contradiction. Hence, $T(v) \in V_{l,k}$. It follows from the induction hypothesis (and from the fact that T is a weight-preserving bijection) that

$$T(\lambda_1 e_{1,1} + \cdots + \lambda_k e_{1,k}) = \alpha_1 e_{l,1} + \cdots + \alpha_k e_{l,k}$$

where $\lambda_k \neq 0 \Rightarrow \alpha_k \neq 0$. Therefore $T(V_{1,k}) = V_{l,k}$. If follows that $T(V_1) \subseteq V_l$; since both have the same dimension, we conclude that they are equal. $\square$

We recall that we defined an action of the symmetric group S_m of permutations of m symbols on V by

$$T_\sigma(v) := v_{\sigma(1)} + v_{\sigma(2)} + \cdots + v_{\sigma(m)}$$

And that we defined an action of $(G_n)^m$ on V by

$$(g_1, g_2, \ldots, g_m)(v_1 + v_2 + \cdots + v_m) = g_1(v_1) + \cdots + g_m(v_m).$$

Lemma 4.4. *Let (V, d_P) be the $m \cdot n$-Rosenbloom–Tsfasman space over $\mathbb{F}_q$. Each symmetry of V that preserves the origin is a product $T_\sigma \circ g$, with σ in S_m and g in $(G_n)^m$.*

Proof. Let T be a symmetry of V, $T(0) = 0$. By the previous result, for each $1 \leq i \leq m$ there is a $k = k(i)$ such that $T(V_i) = V_{k(i)}$ and since T is a bijection, it follows that the map $i \mapsto k(i)$ is a permutation of the set $\{1, \ldots, m\}$. We define $T_\sigma : V \to V$ by

$$T_\sigma(v) := v_{\sigma(1)} + v_{\sigma(2)} + \cdots + v_{\sigma(m)}$$

and then $T = T_\sigma(T_\sigma^{-1}T)$, where $\sigma \in S_m$. Let $g = (T_\sigma^{-1}T)$; clearly $g(V_i) = V_i$, and $g|_{V_i}$ is a symmetry of V_i. Defining $g_i := g|_{V_i}$ we have that $g = (g_1, \ldots, g_n)$ and hence $g \in (G_n)^m$. $\square$

Theorem 4.1. *Let (V, d_P) be the $m \cdot n$-Rosenbloom–Tsfasman space over $\mathbb{F}_q$. The group of symmetries of V is isomorphic to $(G_n)^m \rtimes S_m$.*

Proof. As before, let $G_{m \cdot n}$ be the group of symmetries of V generated by the action of $(G_n)^m$ and S_m. Let T be a symmetry of V and let $v = T(0)$. The translation $S_{-v}(u) := u - v$ is clearly a symmetry of V and $(S_{-v} \circ T)(0) = S_{-v}(v) = 0$ is a symmetry that fixes the origin. Hence, by the previous Lemma, $S_{-v} \circ T \in G_{m \cdot n}$. Consider the decomposition of v on the chain spaces, $v = v_1 + \cdots + v_m$, $v_i \in V_i$; we see that the restriction of S_{-v} to V_i is the translation by $-v_i$, hence a symmetry of V_i. It follows that $S_{-v} \in (G_n)^m \subset G_{m \cdot n}$ and hence, that $T = S_v \circ (S_{-v} \circ T)$ is in $G_{m \cdot n}$ and we conclude that $G_{m \cdot n}$ is the symmetry group of V. By Proposition 4.1, $G_{m \cdot n}$ is isomorphic to $(G_n)^m \rtimes S_m$. $\square$

5. Symmetries of products of Rosenbloom–Tsfasman spaces

As an immediate consequence of the results above, we describe the symmetry groups of all poset spaces (V, d_P) where the connected components of the poset P are chains. Let us consider then a poset P which is the disjoint union of chains of varied lengths. We may write

$$P = \bigcup_{i=1}^{k} P_i,$$

where each P_i is an union of m_i chains of the same length n_i, and $n_i \neq n_j$ if $i \neq j$. Let $N = |P|$, $V = \mathbb{F}_q^N$, and consider the poset space (V, d_P). If $V_i = \{v \in V : supp(v) \subset P_i\}$ then we have the decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

and, just as before, $d_P(u, v) = d_{P_1}(u_1, v_1) + \cdots + d_{P_k}(u_k, v_k)$. Note that $\left(V_i, d_{P_i}\right)$ is isometric to the $m_i \cdot n_i$-Rosenbloom–Tsfasman space. If $G_{m_i \cdot n_i}$ is the symmetry group of each V_i, then the group $G = G_{m_1 \cdot n_1} \times \cdots \times G_{m_k \cdot n_k}$ acts on V by

$$(g_1, g_2, \ldots, g_k)(v_1 + v_2 + \cdots + v_k) = g_1(v_1) + g_2(v_2) + \cdots + g_k(v_k)$$

and this obviously identifies G with a subgroup of the symmetry group of V. On the other hand, the proof of Lemma 4.3 applies in this case, that is, each symmetry T that fixes the origin must take chain subspace onto chain subspace. Since the chains of same length are the components of each P_i, if $T(0) = 0$ then each V_i is invariant under T; hence $T \in G$. From now on we use the same reasoning as in Lemma 4.4 and Theorem 4.1 to conclude that every symmetry of V is an element of G.

Theorem 5.1. *Let P_i be the poset consisting of the disjoint union of m_i chains of the same length n_i for $i = 1, 2, \ldots, k$,*

$$P = \bigcup_{i=1}^{k} P_i$$

and let $V = \mathbb{F}_q^{|P|}$. The symmetry group of the poset space (V, d_P) is isomorphic to the group

$$G_{m_1 \cdot n_1} \times \cdots \times G_{m_k \cdot n_k}.$$

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